The Optimization of the Profit of a Parallel System with Independent Components and Linear Repairing Cost

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Abstract. The parallel system can be regarded as a multi-state system with graduate failure. When the system is not in its perfect state, it can be repaired to some higher level under some cost, in our case, to repair k components costs $C_0 + kC_1$. The objective of this research is to find the optimal repairing policy, so that the system makes the greatest possible profit. The main idea of the optimal solution is based on the analysis of the system performance during periods with a certain length, which allows us to use dynamic programming as optimizing technique. Additionally for the systems with unlimited working time we give a way for computation of the optimal repairing level.

Keywords: Optimal profit, multi-state system, parallel system.

1 Introduction

Consider a parallel system with n independent components, so that each can be in either working or failure state. Some examples of such system are equal machines in a factory that do the same work, computers in the laboratory, buses in a transportation company or n-triple transportation line. The cost to repair a failure component does not always depend only on the bill for individual repairing, but sometimes there are additional penalties that need to be paid like a transportation expenses or some influences on the system as a result of system reconstruction. For that reason, we assume that making a collection of recovers takes constant price C_0 and recovering of individual component takes come expected cost C_1. The whole system can be regarded as a multi-state system, so k effective operating components can be regarded as a system working in level k. During the operation some of the components may fail and we assume that the random variable "time to failure" of each component has exponential distribution with parameter λ. Since the our assumption is that components are independent, one level transition intensities can be regarded as independent Markovian transitions. If in the inspection there are failed components, we may decide to recover some of them. The objective here is to decide on which level it is best to get the system, thereby obtaining the maximal future operating profit.

A similar machine replacement problem is given in [1], where the problem is solved by using dynamics programming, so, here we are led by the same idea. The similar computations of the optimal policy for another type of multi-state systems are given in [3] and [4], where it is found that when the system works long enough there is a level on which the optimal profit is obtained, whenever the system is repaired when it is found under that level. On that line, we concentrate on analyzing the existence of such level.

2 The Optimal Policy for Constant Time Periods

Consider a system that operates *m* time periods with length *T*. During a period of operation some of the components may fail, i.e. the state of the system can become worst. We assume that at the start of each period, we know the state of the system and we must choose to let the system operate one more period in the state it currently is or repair *k* of the components for a cost $C_0 + kC_1$. Also we assume that the expected operating profit each component makes when it is in the working state for a unit of time is known and we will denote it by *C*. The problem we regard is to find the optimal policy for system repairing in order to obtain the benefit of bigger future operating profit.

To solve the problem using dynamics programming we need to identify it optimal substructure. Let $\widetilde{C}_i(m)$ be the expected future optimal profit the system makes in the next *k* periods of length *T*, under assumption that it started in state *i* and at the beginning of the time interval *mT* the failure components are not repaired. By $\hat{C}_i(m)$ we will denote the expected future optimal profit the system makes in the next *k* periods of length *T*, under assumption that it started in state *i*. The problem has the following

Optimal substructure: For all $0 \le k \le n$

$$\widetilde{C}_k(m+1) = \widetilde{C}_k(1) + \sum_{i=0}^{k} \hat{C}_i(m) \binom{k}{i} e^{-i\lambda T} (1-e^{-\lambda T})^{k-i} \tag{1}$$

$$\hat{C}_k(m) = \max(\{\widetilde{C}_k(m)\} \cup \{\widetilde{C}_{k+r}(m) - (C_0 + rC_1) \mid 0 < r \le n-k\}) . \tag{2}$$

Next in this chapter we will show that (2) can be simplified. It is clear that $\forall m, \hat{C}_n(m) = \widetilde{C}_n(m)$, so we are concentrating on computation $\hat{C}_k(m)$ for $k < n$.

The expected profit one component makes for time *T*, if at the beginning of the period it is in failure state (and it is not repairing) is $\widetilde{C}_0(1) = 0$, and if at the beginning of the period it is in working state is equal to

$$\widetilde{C}_1(1) = \int_T^{\infty} CT\lambda e^{-\lambda t} dt + \int_0^T Ct\lambda e^{-\lambda t} dt = \frac{C(1-e^{-\lambda T})}{\lambda} .$$

It is easy to conclude that the expected profit k component makes in time T if at the beginning of the period all of them are in working state is equal to

$$\widetilde{C}_k(1) = \frac{kC(1-e^{-\lambda T})}{\lambda}.$$

We will say that an n-component system is profitable if it is feasible to be repaired when all components are in the failure state. It means that there is a period m and number of components k such that

$$\widetilde{C}_k(m) - (C_0 + kC_1) \geq 0.$$

For an n-component profitable system we have that there is a integer m such that

$$\frac{C(1-e^{-\lambda mT})}{\lambda} = \frac{\widetilde{C}_k(mT)}{k} \geq \frac{C_0}{k} + C_1 > \frac{C_0}{n} + C_1. \quad (3)$$

By $\hat{m}$ we will denote the smallest integer such that (3) holds.

Proposition 2.1 $\forall m \in \boldsymbol{N}^+,\ m < \hat{m}$ and $\forall k = \overline{0,n}$, $\hat{C}_k(m) = \dfrac{kC(1-e^{-\lambda mT})}{\lambda}$.

Proof. The proposition is trivial for $\hat{m} = 1$. Let $\hat{m} > 1$ and $m = 1$. We need to proof that $\dfrac{kC(1-e^{-\lambda T})}{\lambda} > \hat{C}_n(1) - (C_0 + (n-k)C_1)$. Since $\hat{C}_n(1) = \dfrac{nC(1-e^{-\lambda T})}{\lambda}$ we have:

$$\frac{kC(1-e^{-\lambda T})}{\lambda} - \left(\frac{nC(1-e^{-\lambda T})}{\lambda} - (C_0 + (n-k)C_1)\right) = C_0 + (n-k)C_1 - \frac{(n-k)C(1-e^{-\lambda T})}{\lambda}$$

$$> C_0 + (n-k)C_1 - (n-k)\left(\frac{C_0}{n} + C_1\right) = \frac{kC_0}{n} > 0.$$

Suppose that the Proposition holds for all $m < m_1 < \hat{m}$. Using (1) it is easy to prove that $\hat{C}_n(m_1) = \widetilde{C}_n(m_1) = nC(1-e^{-\lambda m_1 T})/\lambda$ and $\widetilde{C}_k(m_1) = kC(1-e^{-\lambda m_1 T})/\lambda$. Again

$$\widetilde{C}_k(m_1) - \left(\hat{C}_n(m_1) - (C_0 + (n-k)C_1)\right) > C_0 + (n-k)C_1 - (n-k)\left(\frac{C_0}{n} + C_1\right) > 0.$$

Let $q = 1 - p$. Using identities $p^r(1-p)^l = p^{r+1}(1-p)^l + p^r(1-p)^{l+1}$ and $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$. we can easy prove the following identity

$$\sum_{i=0}^{k+1} A_i \binom{k+1}{i} p^i q^{k+1-i} - \sum_{i=0}^{k} A_i \binom{k}{i} p^i q^{k-i} = p \sum_{i=0}^{k} \left(A_{i+1} - A_i\right) \binom{k}{i} p^i q^{k-i} \quad (4)$$

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Lemma 2.1 For all positive integers *k* and *m*, such that $m \ge \hat{m}$, $\hat{C}_{k+1}(m) - \hat{C}_k(m) \ge C_1$.

Proof: If $\hat{C}_k(m) = \hat{C}_{k'}(m) - (C_0 + (k'-k)C_1)$, $k' > k$, the Lemma is true since $\hat{C}_{k+1}(m) - \hat{C}_k(m) \ge \left(\hat{C}_{k'}(m) - (C_0 + (k'-(k+1))C_1)\right) - \left(\hat{C}_{k'}(m) - (C_0 + (k'-k)C_1)\right) = C_1$.

Now let $\hat{C}_k(m) = \widetilde{C}_k(m)$. The lemma is true for $m = \hat{m}$ since

$$\hat{C}_{k+1}(\hat{m}) - \hat{C}_k(\hat{m}) > \widetilde{C}_{k+1}(\hat{m}) - \widetilde{C}_k(\hat{m}) = \frac{C(1-e^{-\lambda \hat{m} T})}{\lambda} > \frac{C_0}{n} + C_1 > C_1.$$

Suppose that the lemma is true for all i < m. For m + 1, using (4) and $C(1-e^{-\lambda(\hat{m}-1)T})/\lambda \le C_0/n + C$ we have

$$\begin{aligned}
\hat{C}_{k+1}(m+1) - \hat{C}_k(m+1) &\ge \widetilde{C}_{k+1}(m+1) - \widetilde{C}_k(m+1) \\
&= \frac{C(1-e^{-\lambda T})}{\lambda} + e^{-\lambda T}\sum_{i=0}^{k}\left(\hat{C}_{i+1}(m) - \hat{C}_i(m)\right)\binom{k}{i}e^{-i\lambda T}(1-e^{-\lambda T})^{k-i} \\
&\ge \frac{C(1-e^{-\lambda \hat{m} T})}{\lambda} - \frac{Ce^{-\lambda T}(1-e^{-\lambda(\hat{m}-1)T})}{\lambda} + e^{-\lambda T}C_1 \\
&\ge \frac{C_0}{n} + C_1 - \frac{Ce^{-\lambda T}(1-e^{-\lambda(\hat{m}-1)T})}{\lambda} + e^{-\lambda T}C_1 \\
&\ge \frac{C_0}{n} + C_1 - e^{-\lambda T}\left(\frac{C_0}{n} + C_1\right) + e^{-\lambda T}C_1 \ge C_1.
\end{aligned}$$

Theorem 2.1: Suppose that for some $m \in \mathbf{N}$, there is $k' > k$ such that

$$\hat{C}_k(m) = \hat{C}_{k'}(m) - (C_0 + (k'-k)C_1). \tag{5}$$

and k' is the greatest integer that satisfied (5), then $k' = n$.

Proof: Let $k' < n$, then using Lemma 2.1

$$\begin{aligned}
\hat{C}_{k'+1}(m) - (C_0 + (k'+1-k)C_1) &= \hat{C}_{k'+1}(m) - \hat{C}_{k'}(m) + \hat{C}_{k'}(m) - (C_0 + (k'-k)C_1) - C_1 \\
&\ge C_1 + \hat{C}_k(m) - C_1 = \hat{C}_k(m),
\end{aligned}$$

So k' is not the greatest integer for which (5) holds, which is contradiction. So, $k'=n$.

The last Theorem simplifies the formula (2) to

$$\hat{C}_k(m) = \max\{\widetilde{C}_k(m), \widetilde{C}_n(m) - (C_0 + (n-k)C_1)\}. \tag{6}$$

Theorem 2.2 Let $k < n$ be an integer such that $\hat{C}_k(m) = \hat{C}_n(m) - (C_0 + (n-k)C_1)$. Then for all $k' < k$, $\hat{C}_{k'}(m) = \tilde{C}_n(m) - (C_0 + (n-k')C_1)$.

Proof. $\hat{C}_k(m) = \hat{C}_n(m) - (C_0 + (n-k)C_1)$ implies $\tilde{C}_k(m) < \tilde{C}_n(m) - (C_0 + (n-k)C_1)$ i.e. $\tilde{C}_n(m) - \tilde{C}_k(m) > C_0 + (n-k)C_1$.

For $k' = k - 1$, using the proof of Lemma 1 we have

$$\tilde{C}_n(m) - \tilde{C}_{k-1}(m) = \tilde{C}_n(m) - \tilde{C}_k(m) + \tilde{C}_k(m) - \tilde{C}_{k-1}(m) > C_0 + (n-k)C_1 + C_1.$$

This implies $\tilde{C}_{k-1}(m) < \tilde{C}_n(m) - (C_0 + (n-(k-1))C_1)$, so the Theorem holds for k-1. By induction we have that the theorem holds for all $k' < k$.

The last Theorem tells us that at the beginning of each time interval mT, there is a level $k \le n$, so that for all levels smaller then k the optimal policy is obtained by repairing all the failure components and all the levels bigger and equal to k, the optimal policy is obtained when the failure components are not repaired. We will call this level boundary level for m-th step.

3 The Algorithm for Evaluation of the Optimal Repairing Policy

Using the earlier analysis we can construct an algorithm for evaluation of the optimal repairing policy, that takes O(mn). The psevdocode of the algorithm is

```
Input: C0, C1, C, m, λ, T.
Output: The boundary levels for k-th step, k = 1,m
for k=1 to n do
        Ĉ[k] = 0 ;
        C̃[k] = kC(1 - e^{-λT}) / λ ;
for 1 to m do
     Ĉ[n] = C̃[n] + Σ_{i=0}^{n} Ĉ[i] (k choose i) e^{-iλT} (1 - e^{-λT})^{k-i}
     k=0;
     while Ĉ[k] + Σ_{i=0}^{k} Ĉ[i] (k choose i) e^{-iλT} (1 - e^{-λT})^{k-i} < Ĉ[n] - (C0 + (n-k)C1) do
            Ĉ[k] = Ĉ[n] - (C0 + (n-k)C1) ;
            k=k+1;
     print k
     for j = k to n do Ĉ[j] = Ĉ[j] + Σ_{i=0}^{j} Ĉ[i] (k choose i) e^{-iλT} (1 - e^{-λT})^{j-i} .
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It is interesting that in the most of the experiments we made, for different boundary level for m-th step grows as m grows up, and there is some boundary level for $m \to \infty$. But there are some examples in which for some particular levels m and $m + 1$ the boundary level for m-th step is greater then the boundary level for $m + 1$-th step. Next we give such an example.

Example 3.1 Let $\lambda = 1\ e^{-T} = 0.4545$, $C = 20.18$, $C_0 = 10$, $C_1 = 6$, $n = 2$. For $m = 1$ $\hat{C}_0(1) = \hat{C}_2(1) - (C_0 + 2C_1) = 0.016$ and $\hat{C}_1(1) = \tilde{C}_1(1) = 11.008$ so the boundary level is 0. The boundary level for $m = 2$ is 1 since $\hat{C}_0(2) = \hat{C}_0(2) - (C_0 + 2C_1) = 10.028$ and $\hat{C}_1(2) = \hat{C}_2(2) - (C_0 + C_1) = 16.028$. But, the boundary level for $m = 3$ is 0 again because $\hat{C}_0(3) = \hat{C}_2(3) - (C_0 + 2C_1) = 17.5638$ and $\tilde{C}_1(3) = \tilde{C}_1(3) = 23.7621$. At the next steps, the boundary level remain at 0.

4 Boundary level for a system with unlimited working time

This boundary levels we get in our experiments, inspired us to make an additional analyzing in order to realize the existence of a boundary level when the working time of the system is unknown and we believe that that time is unlimited.

Again we regard a parallel system with n independent components so that the profit each component makes, if it is in the working state for a unit time, is C and the repairing cost for k components cost $C_0 + kC_1$.

Theorem 4.1 Suppose that whenever the system fails to level s, it is repaired to level k. Then the expected mean profit of such system is equal to

$$L_{k,s} = \frac{\lambda(k-s)\left(\frac{C}{\lambda} - \left(\frac{C_0}{k-s} + C_1\right)\right)}{\sum_{i=s+1}^{k} \frac{1}{i}}.$$

Moreover, the maximal expected mean profit is obtained in the case when $k = n$.

Proof. If the system starts with its work in state i, then the expected time to work in level i is equal to $(i\lambda)^{-1}$. The expected transition time from level k to level s is

$$\sum_{i=s+1}^{k} \frac{1}{i\lambda} = \frac{1}{\lambda} \sum_{i=s+1}^{k} \frac{1}{i}.$$

The profit it makes during that time is $\sum_{i=s+1}^{k} \frac{iC}{i\lambda} = \frac{k-s}{\lambda} C.$ To repair it to level k costs $C_0 + (k - s)C_1$. So, the expected mean cost is equal to

$$\frac{\frac{k-s}{\lambda}C-(C_0+(k-s)C_1)}{\frac{1}{\lambda}\sum_{i=s+1}^{k}\frac{1}{i}}=\frac{\lambda(k-s)\left(\frac{C}{\lambda}-\left(\frac{C_0}{k-s}+C_1\right)\right)}{\sum_{i=s+1}^{k}\frac{1}{i}}.$$

In order to proof the second stage of the theorem, we will show that the expected profit when the system is repaired to level $k + 1$ is greater then the expected profit when the system is repaired to level k, whenever it falls to level s, i.e. we will show that the following inequality is true

$$\frac{\lambda(k+1-s)\left(\frac{C}{\lambda}-\left(\frac{C_0}{k+1-s}+C_1\right)\right)}{\sum_{i=s+1}^{k+1}\frac{1}{i}}>\frac{\lambda(k-s)\left(\frac{C}{\lambda}-\left(\frac{C_0}{k-s}+C_1\right)\right)}{\sum_{i=s+1}^{k}\frac{1}{i}}.$$

The last inequality is equivalent to

$$\left(((k-s)+1)\sum_{i=s+1}^{k}\frac{1}{i}\right)\left(\frac{C}{\lambda}-\left(\frac{C_0}{k+1-s}+C_1\right)\right)>\left((k-s)\sum_{i=s+1}^{k+1}\frac{1}{i}\right)\left(\frac{C}{\lambda}-\left(\frac{C_0}{k-s}+C_1\right)\right).$$

It is clear that $\frac{C}{\lambda}-\left(\frac{C_0}{k+1-s}+C_1\right)>\frac{C}{\lambda}-\left(\frac{C_0}{k-s}+C_1\right)$. From the other hand

$$((k-s)+1)\sum_{i=s+1}^{k}\frac{1}{i}=(k-s)\sum_{i=s+1}^{k}\frac{1}{i}+\sum_{i=s+1}^{k}\frac{1}{i}=(k-s)\sum_{i=s+1}^{k+1}\frac{1}{i}+\sum_{i=s+1}^{k}\frac{1}{i}-\frac{k-s}{k+1}>(k-s)\sum_{i=s+1}^{k+1}\frac{1}{i}.$$

The last inequality follows from the fact that for all i, i < k + 1 which implies $\frac{1}{i}>\frac{1}{k+1}$, so

$$\sum_{i=s+1}^{k}\frac{1}{i}-\frac{k-s}{k+1}>\sum_{i=s+1}^{k}\frac{1}{k+1}-\frac{k-s}{k+1}=0.$$

If $\frac{C}{\lambda}\leq\frac{C_0}{n}+C_1$, the system is unprofitable i.e. it is not profitable to repair it. So we will regard only the systems for which $\frac{C}{\lambda}>\frac{C_0}{n}+C_1$. Our goal is to find the level s for which the mean expected profit will be maximal. To do this we need to compare profits P_{n,s_1} and P_{n,s_2} for all $0 \leq s_1, s_2 < n$. The next Theorem gives the boundary for C/λ that under which $P_{n,s} > P_{n,s+r}$, $0 < r < n-s$.

Theorem 4.2 For all $0 \le s < n$ and $0 < r < n-s$, $L_{n,s} > L_{n,s+r}$ if

$$A = \frac{1}{C_0}\left(\frac{C}{\lambda} - C_1\right) < \frac{\sum_{i=s+1}^{s+r}\frac{1}{i}}{(n-s)\sum_{i=s+1}^{s+r}\frac{1}{i} - r\sum_{i=s+1}^{n}\frac{1}{i}} = B_{s,s+r}. \tag{7}$$

Proof. By simple transformation the inequality $L_{n,s} > L_{n,s+r}$ becomes

$$\left(\frac{\sum_{i=s+1}^{n}\frac{1}{i} - \sum_{i=s+r+1}^{n}\frac{1}{i}}{(n-(s+r))\sum_{i=s+1}^{n}\frac{1}{i} - (n-s)\sum_{i=s+r+1}^{n}\frac{1}{i}}\right)C_0 + C_1 > \frac{C}{\lambda}.$$

Using $\sum_{i=s+1}^{n}\frac{1}{i} - \sum_{i=s+r+1}^{n}\frac{1}{i} = \sum_{i=s+1}^{s+r}\frac{1}{i}$ we get $B_{s,s+r}C_0 + C_1 > \frac{C}{\lambda}$, which is equivalent with (7).

Proposition 4.1 $\forall s, r, k$ such that $0 \le s, r, k$ and $s + r + k < n$, $B_{s,s+r} < B_{s,s+(r+k)}$.

Proof: By simple transformation $B_{s,s+r} < B_{s,s+(r+k)}$ becomes

$$\left(\sum_{i=s+1}^{s+r}\frac{1}{i}\right)\left((n-s)\sum_{i=s+1}^{s+r+k}\frac{1}{i} - (r+k)\sum_{i=s+1}^{n}\frac{1}{i}\right) < \left(\sum_{i=s+1}^{s+r+k}\frac{1}{i}\right)\left((n-s)\sum_{i=s+1}^{s+r}\frac{1}{i} - r\sum_{i=s+1}^{n}\frac{1}{i}\right).$$

It is easy to see that after multiplication of the both sides, the first terms will be equal. So this inequality is equivalent with

$$r\sum_{i=s+1}^{s+r+k}\frac{1}{i} < (r+k)\sum_{i=s+1}^{s+r}\frac{1}{i} \Leftrightarrow \sum_{j=s+1}^{s+r}\sum_{i=s+1}^{s+r+k}\frac{1}{i} < \sum_{j=s+1}^{s+r+k}\sum_{i=s+1}^{s+r}\frac{1}{i}$$

$$\Leftrightarrow \sum_{j=s+1}^{s+r}\left(\sum_{i=s+1}^{s+r}\frac{1}{i} + \sum_{i=s+r+1}^{s+r+k}\frac{1}{i}\right) < \sum_{j=s+1}^{s+r}\sum_{i=s+1}^{s+r}\frac{1}{i} + \sum_{j=s+r+1}^{s+r+k}\sum_{i=s+1}^{s+r}\frac{1}{i}.$$

Again we have equal first terms, so $B_{s,s+r} < B_{s,s+(r+k)}$ is equivalent to

$$\sum_{j=s+1}^{s+r}\sum_{i=s+r+1}^{s+r+k}\frac{1}{i} < \sum_{j=s+r+1}^{s+r+k}\sum_{i=s+1}^{s+r}\frac{1}{i} \Leftrightarrow \sum_{j=s+r+1}^{s+r+k}\sum_{i=s+1}^{s+r}\left(\frac{1}{i} - \frac{1}{j}\right) > 0.$$

The last inequality holds because for all *i* and *j*, $i \le s + r < j \Leftrightarrow 1/j < 1/i$.

Proposition 4.2 For all $0 < s < n$, $B_{s-1,s} < B_{s,s+1}$.

Proof: We need to proof that

$$\frac{\frac{1}{s}}{\frac{n-(s-1)}{s}-\sum_{i=s}^{n}\frac{1}{i}} < \frac{\frac{1}{s+1}}{\frac{n-s}{s+1}-\sum_{i=s+1}^{n}\frac{1}{i}},$$

which is equivalent to

$$\frac{n-s}{s(s+1)}-\frac{1}{s}\sum_{i=s+1}^{n}\frac{1}{i}<\frac{n-s+1}{s(s+1)}-\frac{1}{s+1}\sum_{i=s}^{n}\frac{1}{i} \Leftrightarrow -(s+1)\sum_{i=s+1}^{n}\frac{1}{i}<1-s\sum_{i=s}^{n}\frac{1}{i} \Leftrightarrow -\sum_{i=s+1}^{n}\frac{1}{i}<1-1=0.$$

Using last two propositions we can prove the following theorem that characterizes the optimal repairing level.

Theorem 4.1 Let $B_{-1,0}=\frac{1}{n}$, $B_{n,n+1}=\infty$ and $B_{s,s+1}$ are defined as in Theorem 4. Then if $B_{s-1,s}\le A<B_{s,s+1}$, the maximal profit is obtained when all failed components are repaired whenever the parallel system is found at level *s*.

Proof. Using Proposition 1 we have $A<B_{s,s+1}\le B_{s,s+k}$, for all *k*, $1 \le k \le n - s$. From Theorem 3 we have that for all *k*, $1 \le k \le n - s$ $L_{n,s} > L_{n,s+k}$.

From the other side, since $B_{s-1,s}\le A$, from the Proposition 2 we have that for all $1 \le k \le s$, $B_{s-k,s-k+1}\le A$. Using Theorem ? we have that $L_{n,s-k} < L_{n,s-k+1}$, for all $1 \le k \le s$. This imply $L_{n,s} > L_{n,s-k}$, for all $1 \le k \le s$.

From the last two theorems, in order to find the optimal repairing level we need to calculate all B_{s-1}, *s* for $0 \le s \le n$. These numbers grow as *s* grows up, so we need to find the interval in which A belongs. If it is found on the interval ($B_{s-1,s}$, $B_{s,s+1}$), then we will conclude that the maximal mean profit will be obtained if the system is repaired in the moment when it is found at level *s*. These intervals are given in Fig. 1.

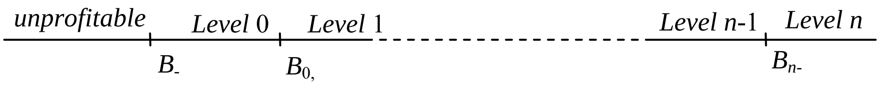

Fig. 1. Decision intervals.

Example 4.1 Consider a system with 4 components. The boundaries are $B_{-1,0}$ = 1/4, B_0, 1 = 12/23, B_1, 2 = 6/5 and B2, 3 = 4. Let C_0=3 and C_1 = 1. Then for C/λ = 1.5, A = 0.167 so unprofitable. For C/λ = 2, A = 0.33 so the optimal repairing level is 0. For C/λ = 6, A = 0.67 so the optimal repairing level is 2. For C/λ = 15, A = 4.67 so the optimal repairing level is 3.

The same result we obtained experimentally. The operation of such system during the time T1, much greater than T, was simulated. Whenever the system enters the specific level *k*, it is repaired to the level s, ∀s > k. The optimal profit was always received for *s* = *n* and the optimal level matches with the theoretical results.

We can design an O(*n*) algorithm for calculating the boundaries and finding the optimal repairing level.

```
Input: C0, C1, C, n, λ.
Output: if the system is profitable, the optimal repairing
level, else the message that the system is not profitable
A=(1/C0)(C/λ- C1)
if A < 1/n then print "the system is unprofitable" else
  S = 1/n
  s = n-1
  while A<=(1/(s+1))/( (n-s)/(s+1)-S) and s>=0 do
    S = S+1/(s+1);
    s=s-1;
if s ≠ -1 then print "the optimal repairing level is" s+1.
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5 Conclusion

This paper deals with operating process of a parallel *n*-component system. The objective is to find the level to which the system with *k* efficiently operating components at the beginning of each considering time needs to be repaired, or to make decision to left it at the current state, in order to obtain the maximal future operating profit. We regard two types of systems. For the first type we assume that at the start of each period we know its state and only in that moment we are able to repair some of its components. For such systems, we showed that in order to obtain the optimal future operation profit, at the beginning of each period we need to make decision only between two choices, to repair all failure components or to left the system operate one more period in its current state. Moreover, it is shown that there is a boundary level under which the optimal policy is obtained if we leave the system in its current state. The current state for the second type of systems is known at any moment during their operation, which also allows us to make decision to repair its components in each moment. It is shown that there is a level *k* under which it is not profitable to fail, i.e. that the optimal mean profit is obtained if all failure components were repaired whenever the systems takes that level *k*.

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