Some fixed point theorems for cyclic contractions in ultra metric spaces

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Abstract: In this paper we study some fixed point theorems in an uniformly convex Banach space. We see these results for some contract mappings with cyclic operators of Banach type and Kannan type. We give some results on convergence of Picard operators on a spherically complete ultra metric space. Our intention is to give some existence results for approximation of the fixed points for cyclic contractions using comparison functions that can be used in algorithms.

Keywords: fixed point, ultra metric space, Picard operator, cyclic contraction, comparison function.

1. Introduction and Preliminaries

One of the most important results used in functional analysis is the well-known Banach's contraction which in 1922 asserts that:

If (X,d) is a complete metric space and $T : X \to X$ is a mapping such that

$$d(T(x),T(y)) \le \lambda d(x,y)$$

for all $x, y \in X$ and some $\lambda \in [0,1)$ then T has a unique fixed point in X.

Definition 1. Let (X,d) be a ultra metric space. A mapping $T : X \to X$ is called a ϕ-contraction if there exists a comparison function $\phi : R^+ \to R^+$ such that $d(T(x),T(y)) \le \varphi(d(x,y))$ for all $x, y \in X$.

Definition 2 Let (X,d) be a ultra metric space, m a positive integer $A_1,...,A_m$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$ an operator $T : Y \to Y$ is called a cyclic ϕ-contraction if

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T

(ii) There exists a (c)-comparison function $\phi : R^{+} \to R^{+}$ such that $d(T(x),T(y)) \le \varphi(d(x,y))$
for any $x \in A_i$, $y \in A_{i+1}$ where $A_{m+1} = A_i$

Definition 3. A function $\phi : R^{+} \to R^{+}$ is called a (c)-comparison function if it satisfies:
(i) ϕ is monotone increasing;
(ii) there exist $k_0 \in \mathrm{N}, a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$\varphi^{k+1}(t) \le \alpha\varphi^{k}(t) + v_k$ for $k \ge k_0$ and any $t \in R_{+}$. Let us denote this family with $\mathcal{F}$

2. Main Results

Theorem 4. Let (X,d) be a ultra metric space, m a positive integer $A_1,...,A_m$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$, a (c)-comparison function $\phi : R^{+} \to R^{+}$, an operator $T : Y \to Y$
Assume that
(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T
(ii) T is a cyclic ϕ-contraction.

Then T has a unique fixed point $x^{*} \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $\{x_n\}$ converges to x^{*} for any initial point $x_0 \in Y$.

Now we will prove that the Picard iteration converges to x^{*} for any initial point $x \in Y$. Let $x \in Y = \bigcup_{i=1}^{m} A_i$, there exists $i_0 = \{1,...,m\}$ such that $x_0 \in A_{i0}$. As $x^{*} \in \bigcap_{i=1}^{m} A_i$ it follows that $x^{*} \in A_{i_0+1}$ as well. Then we obtain: $d(T(x),T(x^{*})) \le \varphi(d(x,x^{*}))$

By induction, it follows that: $d(T^{n}(x),x^{*}) \le \varphi^{n}(T(x,x^{*}))$ $n \ge 0$

Since $d(x^*, x^*) \le d(T^n(x), x^*)$ we have $d(x^*, x^*) \le \varphi^n(d(x, x^*))$

Now letting $n \to \infty$ and supposing $x \ne x^*$ we have $d(x^*, x^*) = \lim_{n\to\infty} d(T^n(x), x^*) = 0$

Definition 5. Let (X, d) be an ultra metric space, m be a positive integer, $A_1, A_2, ..., A_m$ be nonempty subsets of X and $X = \bigcup_{i=1}^{m} A_i$ .An operator $T : X \to X$ is a *cyclic weak* $(\varphi - \psi)$*-contraction* if

(i) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T

(ii) $\phi(d(Tx, Ty)) \le \phi(d(x, y)) - \psi(d(x, y))$ for any $x \in A_i$

$y \in A_{i+1}, i = 1,2,..., m$, where

$A_{i+1} = A_1$ and. $\phi, \psi \in F$

An important result based on Karapinar, Sadarangani is the following.

Theorem 6. Let (X, d) be a complete metric space, m be a positive integer, $A_1, A_2, ..., A_m$ be nonempty subsets of X and $X = \bigcup_{i=1}^{m} A_i$. Let $T : X \to X$ be *a cyclic* $(\varphi - \psi)$*-contraction* with $\phi, \psi \in \mathcal{F}$. Then T has a unique fixed point $z \in \cap_{i=1}^{m} A_i$

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